An Analysis of a Hybrid-Mode in a Twisted Rectangular Waveguide

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Abstract —Analytic expressions of electromagnetic fields for the dominant hybrid-mode in a twisted rectangular waveguide are obtained. The fields exactly satisfy the boundary conditions on the guide walls in a helicoidal shape.

By expanding these expressions for the fields in terms of the eigenfunctions of a straight waveguide, the hybrid-mode is found to be composed of a fundamental TE_{10}-mode component, accompanied with TE_{01}, TM_{21}, TE_{21}, and TE_{03} modes, as successive higher order components. The result of the modal power calculation reveals that there exists a frequency at which the transmitting power carried in the cross-polarized TE_{01}-mode component just vanishes.

As a limiting case of the twisted waveguide, fields in a twisted strip line are discussed also, and the existence of a propeller-like equiphase surface is shown.

I. Introduction

In 1955, Lewin [1] published a paper on the analysis of the twisted waveguides and, more recently, he also discussed the degeneracy problem in twisted square waveguides [2]. Although the first-order field expansion was obtained in the process of Lewin's work [1]–[4], the electromagnetic fields were assumed to be of the TE-mode type whose electric field lies entirely on the plane perpendicular to the guide axis. It should be mentioned, however, that the TE-mode fields do not exactly satisfy all the boundary conditions throughout the perfectly conducting guidewall surfaces in helicoidal shape.

This paper is concerned with a boundary value problem of the twisted waveguide. The exact boundary conditions, in terms of the local oblique base vectors, are presented in Section II. The analytic expressions for hybrid-mode fields that exactly satisfy the boundary conditions are given in Section III. The dominant-mode fields of the hybrid type obtained here are rather involved. The mode fields are expanded in the form of a modal series in Section IV. Furthermore, electromagnetic fields in a twisted strip line are also discussed in Section V as a limiting case of the twisted rectangular waveguide.

II. Boundary Conditions

A uniformly twisted rectangular waveguide in fixed Cartesian coordinates (x, y, and z) is illustrated in Fig. 1(a). Focusing our attention on the helicoidal wall surfaces, we intend to find the equations for these boundary surfaces as simply as possible mathematically. We introduce here the twisted coordinates (X, Y, and Z) in which axes X and Y rotate simultaneously with the rotation of the cross section. They are related to the fixed Cartesian coordinates by the equations

$$X = x\cos\alpha z + y\sin\alpha z$$
$$Y = -x\sin\alpha z + y\cos\alpha z$$
$$Z = z \quad (1)$$

where α is called the twist constant in rad/m [5]. In terms of the twisted coordinates, the four helicoidal surfaces can be expressed in simpler form

$$X = \pm\frac{a}{2}, \quad \text{for narrow walls}$$
$$Y = \pm\frac{b}{2}, \quad \text{for wide walls} \quad (2)$$

where a and b are cross-sectional dimensions of wide and narrow walls, respectively.

Fig. 1(b) shows two sets of the local oblique base vectors known as the unitary and the reciprocal unitary vectors associated with the point P. The unitary ($\boldsymbol{a}_X$, $\boldsymbol{a}_Y$, and $\boldsymbol{a}_Z$) are related to the unit vectors ($\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$) of the Cartesian

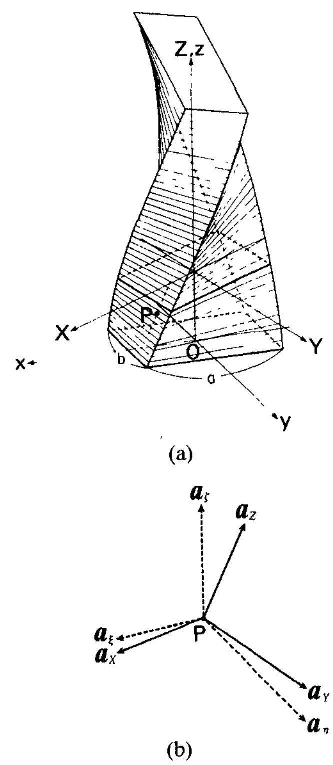

Fig. 1. Twisted rectangular waveguide and coordinate systems. (a) Twisted coordinates (X, Y, and Z) and fixed Cartesian coordinates (x, y, and z) having the same origin 0. (b) Local oblique base vectors at a point P.

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coordinate system by

$$\begin{aligned} \boldsymbol{a}_X &= \boldsymbol{i}\cos\alpha z + \boldsymbol{j}\sin\alpha z \\ \boldsymbol{a}_Y &= -\boldsymbol{i}\sin\alpha z + \boldsymbol{j}\cos\alpha z \\ \boldsymbol{a}_Z &= -\alpha y\boldsymbol{i} + \alpha x\boldsymbol{j} + \boldsymbol{k}. \end{aligned} \tag{3}$$

It must be noted that the unitary vectors are not necessarily of unit length [6], as in the case of $\boldsymbol{a}_Z$ in (3). The reciprocal unitary vectors ($\boldsymbol{a}_\xi$, $\boldsymbol{a}_\eta$, and $\boldsymbol{a}_\zeta$) are defined in order that the following orthogonality relations may hold:

$$\begin{aligned} \boldsymbol{a}_\xi &= \boldsymbol{a}_Y \times \boldsymbol{a}_Z & \boldsymbol{a}_X &= \boldsymbol{a}_\eta \times \boldsymbol{a}_\zeta \\ \boldsymbol{a}_\eta &= \boldsymbol{a}_Z \times \boldsymbol{a}_X & \boldsymbol{a}_Y &= \boldsymbol{a}_\zeta \times \boldsymbol{a}_\xi \\ \boldsymbol{a}_\zeta &= \boldsymbol{a}_X \times \boldsymbol{a}_Y & \boldsymbol{a}_Z &= \boldsymbol{a}_\xi \times \boldsymbol{a}_\eta. \end{aligned} \tag{4}$$

Let the guide walls be of infinite conductivity. Then, the electric- and magnetic-field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ must satisfy the following boundary conditions:

$$\begin{aligned} &\boldsymbol{E} \times \boldsymbol{a}_\xi = 0 \quad \boldsymbol{H}\cdot\boldsymbol{a}_\xi = 0, \quad \text{for narrow walls} \\ &\boldsymbol{E} \times \boldsymbol{a}_\eta = 0 \quad \boldsymbol{H}\cdot\boldsymbol{a}_\eta = 0, \quad \text{for wide walls} \end{aligned} \tag{5}$$

where the base vectors $\boldsymbol{a}_\xi$ and $\boldsymbol{a}_\eta$ are, respectively, perpendicular to the narrow- and wide-wall surfaces.

In order to obtain more convenient expressions of these boundary conditions, we resolve the vector $\boldsymbol{E}$ into its covariant components (E_ξ, E_η, and E_ζ), and the vector $\boldsymbol{H}$ into its contravariant components (H_X, H_Y, and H_Z).

Then we have

$$\begin{aligned} \boldsymbol{E} &= E_\xi \boldsymbol{a}_\xi + E_\eta \boldsymbol{a}_\eta + E_\zeta \boldsymbol{a}_\zeta \\ \boldsymbol{H} &= H_X \boldsymbol{a}_X + H_Y \boldsymbol{a}_Y + H_Z \boldsymbol{a}_Z \end{aligned} \tag{6}$$

and they are also related to the Cartesian components by the following equations:

$$\begin{aligned} E_\xi &= E_x \cos\alpha z + E_y \sin\alpha z \\ E_\eta &= -E_x \sin\alpha z + E_y \cos\alpha z \\ E_\zeta &= -\alpha y E_x + \alpha x E_y + E_z \end{aligned} \tag{7a}$$

$$\begin{aligned} H_X &= H_x \cos\alpha z + H_y \sin\alpha z + \alpha Y H_z \\ H_Y &= -H_x \sin\alpha z + H_y \cos\alpha z - \alpha X H_z \\ H_Z &= H_z. \end{aligned} \tag{7b}$$

Substituting (6) into (5), we can finally obtain the boundary conditions in terms of the field components as follows:

$$\begin{aligned} &E_\eta = E_\zeta = 0 \quad H_X = 0, \quad \text{for } X = \pm\frac{a}{2} \\ &E_\xi = E_\zeta = 0 \quad H_Y = 0, \quad \text{for } Y = \pm\frac{b}{2}. \end{aligned} \tag{8}$$

It must be noted here that the field components in these expressions are represented on the basis of the local oblique base vectors, and E_Z and H_Z components do not vanish on the guide walls.

III. Perturbational-Field Solutions

In order to obtain the field solutions for twisted waveguides of a hybrid type, we start from Maxwell's equations. At first, we express Maxwell's equations in the twisted coordinates and eliminate terms for the magnetic-field components. After rather laborious, but straightforward, calculations similar to those in [5], we obtain the following three sets of simultaneous equations for the electric-field components:

$$\begin{aligned} &\frac{\partial^2 E_\xi}{\partial Y^2} + \frac{\partial^2 E_\xi}{\partial Z^2} + k^2 E_\xi - \frac{\partial^2 E_\eta}{\partial X \partial Y} - \frac{\partial^2 E_\zeta}{\partial X \partial Z} \\ &\quad - \alpha\left[\left(2X\frac{\partial^2 E_\xi}{\partial Y \partial Z} - Y\frac{\partial^2 E_\xi}{\partial X \partial Z}\right) - \left(X\frac{\partial^2 E_\eta}{\partial X \partial Z} - \frac{\partial E_\eta}{\partial Z}\right)\right. \\ &\quad \left. + \left(Y\frac{\partial^2 E_\zeta}{\partial X^2} - X\frac{\partial^2 E_\zeta}{\partial X \partial Y} - \frac{\partial E_\zeta}{\partial Y}\right)\right] \\ &\quad - \alpha^2\left[-\left(X^2\frac{\partial^2 E_\xi}{\partial Y^2} - XY\frac{\partial^2 E_\xi}{\partial X \partial Y}\right)\right. \\ &\quad \left. + \left(X^2\frac{\partial^2 E_\eta}{\partial X \partial Y} - XY\frac{\partial^2 E_\eta}{\partial X^2}\right)\right] = 0 \end{aligned} \tag{9a}$$

$$\begin{aligned} &\frac{\partial^2 E_\eta}{\partial X^2} + \frac{\partial^2 E_\eta}{\partial Z^2} + k^2 E_\eta - \frac{\partial^2 E_\xi}{\partial X \partial Y} - \frac{\partial^2 E_\zeta}{\partial Y \partial Z} \\ &\quad + \alpha\left[\left(2Y\frac{\partial^2 E_\eta}{\partial X \partial Z} - X\frac{\partial^2 E_\eta}{\partial Y \partial Z}\right) - \left(Y\frac{\partial^2 E_\xi}{\partial Y \partial Z} - \frac{\partial E_\xi}{\partial Z}\right)\right. \\ &\quad \left. + \left(X\frac{\partial^2 E_\zeta}{\partial Y^2} - Y\frac{\partial^2 E_\zeta}{\partial X \partial Y} - \frac{\partial E_\zeta}{\partial X}\right)\right] \\ &\quad - \alpha^2\left[-\left(Y^2\frac{\partial^2 E_\eta}{\partial X^2} - XY\frac{\partial^2 E_\eta}{\partial X \partial Y}\right)\right. \\ &\quad \left. + \left(Y^2\frac{\partial^2 E_\xi}{\partial X \partial Y} - XY\frac{\partial^2 E_\xi}{\partial Y^2}\right)\right] = 0 \end{aligned} \tag{9b}$$

$$\begin{aligned} &\frac{\partial^2 E_\zeta}{\partial X^2} + \frac{\partial^2 E_\zeta}{\partial Y^2} + k^2 E_\zeta - \frac{\partial^2 E_\xi}{\partial X \partial Z} - \frac{\partial^2 E_\eta}{\partial Y \partial Z} \\ &\quad - \alpha\left[\left(X\frac{\partial^2 E_\zeta}{\partial Y \partial Z} - Y\frac{\partial^2 E_\zeta}{\partial X \partial Z}\right) + \left(Y\frac{\partial^2 E_\xi}{\partial Z^2} - 2\frac{\partial E_\xi}{\partial Y}\right)\right. \\ &\quad \left. - \left(X\frac{\partial^2 E_\eta}{\partial Z^2} - 2\frac{\partial E_\eta}{\partial X}\right)\right] \\ &\quad - \alpha^2\left[-\left(Y^2\frac{\partial^2 E_\zeta}{\partial X^2} + X^2\frac{\partial^2 E_\zeta}{\partial Y^2} - X\frac{\partial E_\zeta}{\partial X}\right.\right. \\ &\quad \left. - Y\frac{\partial E_\zeta}{\partial Y} - 2XY\frac{\partial^2 E_\zeta}{\partial X \partial Y}\right) \\ &\quad + \left(Y^2\frac{\partial^2 E_\xi}{\partial X \partial Z} - XY\frac{\partial^2 E_\xi}{\partial Y \partial Z} - X\frac{\partial E_\xi}{\partial Z}\right) \\ &\quad \left. + \left(X^2\frac{\partial^2 E_\eta}{\partial Y \partial Z} - XY\frac{\partial^2 E_\eta}{\partial X \partial Z} - Y\frac{\partial E_\eta}{\partial Z}\right)\right] = 0 \end{aligned} \tag{9c}$$

where $k^2 = \omega^2 \epsilon\mu$ and $\omega = 2\pi f$.

In the boundary conditions (8) as well as the simultaneous equations (9a) to (9c), if the twist constant α is put to zero, then these relations are naturally reduced to those for straight waveguides. Keeping this nature in mind, and paying attention to the form of fields for the dominant TE_{10} mode in the straight waveguide, we presume the solutions of electric components in the perturbed form as follows:

$$E_\xi = \alpha\Phi_\xi^{(1)}\exp(-j\beta_T Z)$$
$$E_\eta = \left(\cos\frac{\pi X}{a} + \alpha\Phi_\eta^{(1)}\right)\exp(-j\beta_T Z)$$
$$E_\zeta = \alpha\Phi_\zeta^{(1)}\exp(-j\beta_T Z) \tag{10}$$

where α is assumed to be a small quantity. Since the field solutions of the first-order approximation are of interest, the higher order terms are neglected here. The sign of α cannot affect the phase constant β_T, as pointed out by Lewin [1]. Thus, in the first approximation, we write

$$\beta_T = \beta_g = \sqrt{k^2 - \frac{\pi^2}{a^2}} \tag{11}$$

where β_g is the unperturbed phase constant in the straight waveguide of the same cross section.

Considering the boundary conditions (8), we expand the scalar functions $\Phi_\xi^{(1)}$, $\Phi_\eta^{(1)}$, and $\Phi_\zeta^{(1)}$ in the form of the double Fourier series

$$\Phi_\xi^{(1)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}^{(1)}\cos m\theta \sin n\psi$$
$$\Phi_\eta^{(1)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}^{(1)}\sin m\theta \cos n\psi$$
$$\Phi_\zeta^{(1)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}^{(1)}\sin m\theta \sin n\psi \tag{12}$$

where a change of variables

$$\theta = \frac{\pi}{a}\left(X + \frac{a}{2}\right), \qquad 0 \leqslant \theta \leqslant \pi$$
$$\psi = \frac{\pi}{b}\left(Y + \frac{b}{2}\right), \qquad 0 \leqslant \psi \leqslant \pi \tag{13}$$

has been made.

Substituting (10) together with (12) into (9a) to (9c), equating the coefficients of α to zero, and integrating over the cross section, we have three simultaneous equations for the unknown coefficients $A_{mn}^{(1)}$, $B_{mn}^{(1)}$, and $C_{mn}^{(1)}$ which can be solved in a straightforward manner.

The solutions obtained directly from these equations are in the form of an infinite series with double summations as in (12). We transform these solutions into a more closed form with the aid of partial fraction expansions, as well as the formulas for Fourier series expansion.

This leads to the following results:

$$\Phi_\xi^{(1)} = j\frac{\beta_g a^2}{\pi^2}\left\{\left(\theta - \frac{\pi}{2}\right)\cos\theta - \sin\theta + \frac{4}{\pi}\sum_{m=0,2,\cdots}^{\infty} \cdot \frac{\epsilon_m \cos m\theta}{(m^2-1)^2} \cdot \frac{\cosh\left[\phi\sqrt{m^2-1}\left(\frac{2}{\pi}\psi - 1\right)\right]}{\cosh\left(\phi\sqrt{m^2-1}\right)}\right\}$$

$$\Phi_\eta^{(1)} = j\frac{\beta_g a^2}{\pi^2}\left\{\frac{b}{a}\left(\theta - \frac{\pi}{2}\right)\left(\psi - \frac{\pi}{2}\right)\sin\theta + \frac{16}{\pi}\sum_{m=2,4,\cdots}^{\infty}\frac{m\sin m\theta}{\left(\sqrt{m^2-1}\right)^5} \cdot \frac{\sinh\left[\phi\sqrt{m^2-1}\left(\frac{2}{\pi}\psi - 1\right)\right]}{\cosh\left(\phi\sqrt{m^2-1}\right)}\right\}$$

$$\Phi_\zeta^{(1)} = \frac{a}{\pi}\left\{\left(\theta - \frac{\pi}{2}\right)\sin\theta + \frac{8}{\pi}\sum_{m=2,4,\cdots}^{\infty}\frac{m\sin m\theta}{(m^2-1)^2} \cdot \frac{\cosh\left[\phi\sqrt{m^2-1}\left(\frac{2}{\pi}\psi - 1\right)\right]}{\cosh\left(\phi\sqrt{m^2-1}\right)}\right\} \tag{14}$$

where

$$\phi = \pi b/2a$$
$$\epsilon_m = 1, \qquad \text{for } m = 0,$$

and

$$\epsilon_m = 2, \qquad \text{for } m \neq 0.$$

Although these functions include involved summation terms, the series is found to be convergent so rapidly that the first two terms ($m = 0, 2$) are sufficient for conventional applications ($0.4 \lesssim b/a \lesssim 0.5$). Furthermore, these functions have derivatives of all orders and the series is differentiable term-by-term with respect to θ and ψ.

Substitution of (14) into (10) gives the covariant components of the electric field in analytic form. Contravariant components of the magnetic field can be derived by using Maxwell's equations

$$H_X = -\frac{1}{j\omega\mu}\left(\frac{\partial E_\zeta}{\partial Y} - \frac{\partial E_\eta}{\partial Z}\right)$$
$$H_Y = -\frac{1}{j\omega\mu}\left(\frac{\partial E_\xi}{\partial Z} - \frac{\partial E_\zeta}{\partial X}\right)$$
$$H_Z = -\frac{1}{j\omega\mu}\left(\frac{\partial E_\eta}{\partial X} - \frac{\partial E_\xi}{\partial Y}\right). \tag{15}$$

Up to this step, we obtained both the electric- and the magnetic-field solutions that satisfy Maxwell's equations to the first approximation. So far, we have imposed the boundary conditions only on the electric fields. However, the magnetic-field components given in (15) automatically satisfy their boundary conditions given in (8), because they hold for perfectly conducting boundaries in general.

As shown in the elementary relations (7a) and (7b), the electromagnetic fields obtained here have both electric and magnetic axial components. Thus the dominant mode in the twisted waveguide is found to be of the hybrid type as mentioned before.

The perturbed phase constant β_T associated with the dominant hybrid-mode wave can be obtained as the second-order approximation based on these analytic field solutions. It will be discussed later [7].

IV. Modal Series Expansion

In order to clarify the hybrid-mode fields of (10) and (15), we take the field patterns at $Z=0$, and transform them into the expressions in Cartesian components by the aid of (7a) and (7b). Next, we expand these components in terms of the eigenfunctions of the straight waveguide as follows:

$$E_x = -\sqrt{\frac{2}{ab}}\left(V_{[01]}\sin\psi + V_{[03]}\sin 3\psi\right) + \frac{2}{U_{21}}\left(\frac{2V_{(21)}}{a} - \frac{V_{[21]}}{b}\right)\cos 2\theta \sin\psi + \cdots$$

$$E_y = V_{[10]}\sqrt{\frac{2}{ab}}\sin\theta + \frac{2}{U_{21}}\left(\frac{V_{(21)}}{b} + \frac{2V_{[21]}}{a}\right)\sin 2\theta\cos\psi + \cdots$$

$$E_z = -\frac{1}{j\omega\epsilon}\cdot\frac{2\pi}{ab}U_{21}I_{(21)}\sin 2\theta\sin\psi + \cdots$$

$$H_x = -I_{[10]}\sqrt{\frac{2}{ab}}\sin\theta - \frac{2}{U_{21}}\left(\frac{I_{(21)}}{b} + \frac{2I_{[21]}}{a}\right)\sin 2\theta\cos\psi + \cdots$$

$$H_y = -\sqrt{\frac{2}{ab}}\left(I_{[01]}\sin\psi + I_{[03]}\sin 3\psi\right) + \frac{2}{U_{21}}\left(\frac{2I_{(21)}}{a} - \frac{I_{[21]}}{b}\right)\cos 2\theta\sin\psi + \cdots$$

$$H_z = -\frac{1}{j\omega\epsilon}\cdot\frac{\pi}{a}\sqrt{\frac{2}{ab}}\left[V_{[10]}\cos\theta + \frac{a}{b}\cdot\left(V_{[01]}\cos\psi + 3V_{[03]}\cos 3\psi\right) + \sqrt{\frac{2a}{b}}U_{21}V_{[21]}\cos 2\theta\cos\psi + \cdots\right] \tag{16}$$

where

$$U_{21} = \sqrt{\frac{4b}{a} + \frac{a}{b}}.$$

The notations V and I with double subscripts are used as the modal expansion coefficients, where brackets [ ] are used for TE modes and parentheses ( ) are used for TM modes.

The modal expansion coefficients can easily be determined by means of a conventional technique for double Fourier series expansion. The final results are listed in Table I. As an example, the numerical values of the model expansion coefficients in a normalized form are illustrated in Fig. 2, where the cross-sectional dimensions $a = 22.9$ mm, $b = 10.2$ mm, and the twist constant $\alpha = 90°/5.7$ cm (normalized as $\bar{\alpha} = \alpha a/2\pi = 0.1$) are assumed. The frequency range is restricted in the region where only the dominant mode can propagate.

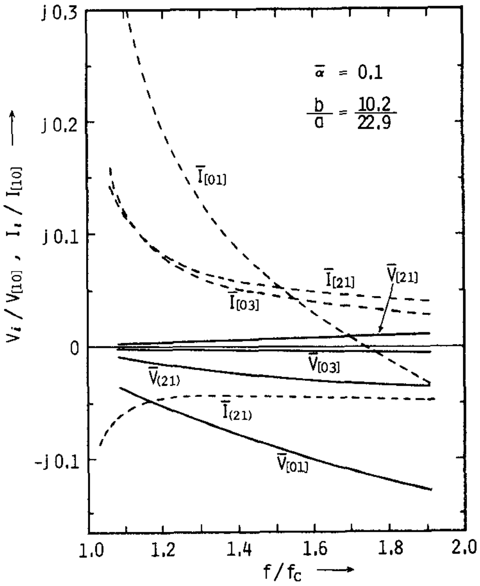

Fig. 2. The ratio of an individual modal expansion coefficient to that of a fundamental TE_{10} component as a function of normalized frequency, where f_c is the cutoff frequency in the straight waveguide.

It is easily shown that the first few terms of model components, namely, TE_{10}, TE_{01}, TM_{21}, TE_{21}, and TE_{03}, yield a very good approximation to the hybrid-mode fields. As a matter of course, the most significant contribution is made by the fundamental TE_{10} components. This means practically simultaneous rotation of the field pattern along the twist is seen in the waveguide.

The contribution of the higher order modal components are obtained from the calculation of modal power P_i given by

$$P_i = R_e\left(V_i I_i^*\right) \tag{17}$$

where subscript i indicates the respective higher order modes.

Fig. 3 shows the frequency dependence of the individual modal power normalized with $P_{[10]}$ calculated by using (17) and Table I. A remarkable behavior of the cross-polarized TE_{01} component is observed in the figure. The power carried in this modal component varies from negative to positive with increasing frequency. The minus sign of the power accounts for, as known in general, the backward wave property of the hybrid wave.

Because of this fact, there exists a specific frequency f_0 given by

$$\frac{f_0}{f_c} = \sqrt{\frac{1}{2}\left[\left(\frac{a}{b}\right)^2 + 1\right]} \tag{18}$$

at which the power carried in the TE_{01}-modal component just vanishes.

The TE_{01}-modal power also depends critically on the waveguide aspect ratio b/a as shown in Fig. 4.

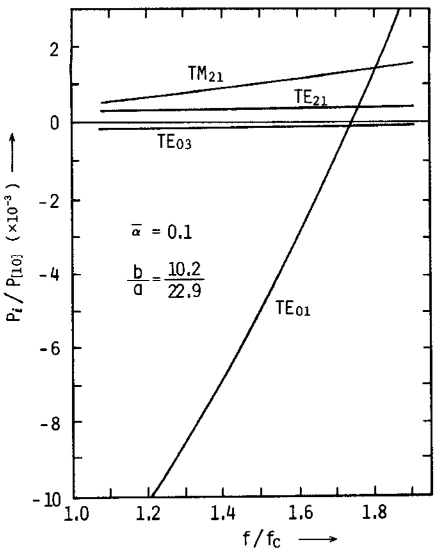

Fig. 3. Frequency dependence of normalized power carried by an individual modal component.

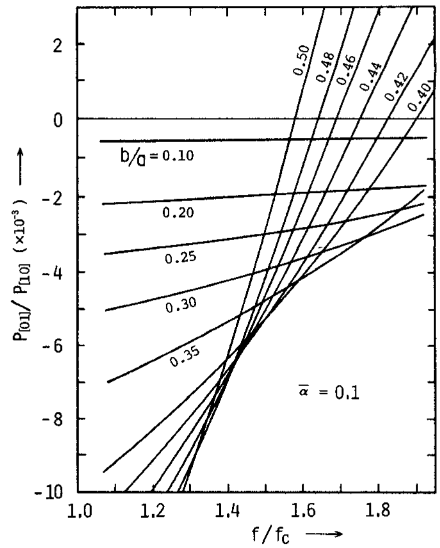

Fig. 4. Effect of the ratio b/a on the frequency dependence of the cross-polarized TE_{01}-modal power.

TABLE I
MODAL EXPANSION COEFFICIENTS FOR THE DOMINANT HYBRID MODE.

mode	$\overline{V}$	$\overline{I}$
TE_{10}	$\overline{V}_{[10]} = 1 \quad \left(V_{[10]} = \sqrt{\frac{ab}{2}}. \right)$	$\overline{I}_{[10]} = 1 \quad \left(I_{[10]} = \frac{\beta g}{\omega\mu} \sqrt{\frac{ab}{2}} \right)$
TE_{01}	$\overline{V}_{[01]} = -j\overline{\alpha} \frac{32}{\pi^2} \sqrt{\overline{f}^2-1} \left(\frac{b}{a}\right)^2 \frac{1}{1-\left(\frac{b}{a}\right)^2}$	$\overline{I}_{[01]} = j\overline{\alpha} \frac{16}{\pi^2} \frac{1}{\sqrt{\overline{f}^2-1}} \frac{1-\left(\frac{b}{a}\right)^2(2\overline{f}^2-1)}{1-\left(\frac{b}{a}\right)^2}$
TE_{03}	$\overline{V}_{[03]} = -j\overline{\alpha} \frac{32}{3\pi^2} \sqrt{\overline{f}^2-1} \left(\frac{b}{a}\right)^2 \frac{1}{9-\left(\frac{b}{a}\right)^2}$	$\overline{I}_{[03]} = j\overline{\alpha} \frac{16}{3\pi^2} \frac{1}{\sqrt{\overline{f}^2-1}} \frac{9-\left(\frac{b}{a}\right)^2(2\overline{f}^2-1)}{9-\left(\frac{b}{a}\right)^2}$
TE_{21}	$\overline{V}_{[21]} = j\overline{\alpha} \frac{32\sqrt{2}}{9\pi^2} \sqrt{\overline{f}^2-1} \left(\frac{b}{a}\right)^2 \frac{12\left(\frac{b}{a}\right)^2-1}{\sqrt{1+4\left(\frac{b}{a}\right)^2\left[1+3\left(\frac{b}{a}\right)^2\right]}}$	$\overline{I}_{[21]} = j\overline{\alpha} \frac{16\sqrt{2}}{9\pi^2} \frac{1}{\sqrt{\overline{f}^2-1}} \frac{5-\left(\frac{b}{a}\right)^2(5-2\overline{f}^2)\left[12\left(\frac{b}{a}\right)^2-1\right]}{\sqrt{1+4\left(\frac{b}{a}\right)^2\left[1+3\left(\frac{b}{a}\right)^2\right]}}$
TM_{21}	$\overline{V}_{(21)} = -j\overline{\alpha} \frac{64\sqrt{2}}{9\pi^2} \sqrt{\overline{f}^2-1} \left(\frac{b}{a}\right) \frac{1}{\sqrt{1+4\left(\frac{b}{a}\right)^2\left[1+3\left(\frac{b}{a}\right)^2\right]}}$	$\overline{I}_{(21)} = -j\overline{\alpha} \frac{64\sqrt{2}}{9\pi^2} \frac{\overline{f}^2}{\sqrt{\overline{f}^2-1}} \left(\frac{b}{a}\right) \frac{1}{\sqrt{1+4\left(\frac{b}{a}\right)^2\left[1+3\left(\frac{b}{a}\right)^2\right]}}$

$\overline{V} = \frac{V_i}{V_{[10]}}$, $\overline{I} = \frac{I_i}{I_{[10]}}$, $\overline{\alpha} = \frac{\alpha a}{2\pi}$, $\overline{f} = \frac{f}{f_c}$, f_c = cut off frequency in the straight waveguide.

V. TWISTED STRIP LINE

The central portion of the twisted rectangular waveguide may be considered as a twisted strip line if its wide walls extend to infinity. Such a structure is useful in suggesting a physical picture of rather complicated hybrid fields in the twisted rectangular waveguide. In this section, we shall derive the expressions for the electromagnetic fields of TEM type in the twisted strip line transformed from the twisted waveguide.

Let us now return to (14) in Section III, and concentrate our discussion on the hyperbolic functions. Let ϕ be sufficiently small such that the power series expansion up to the terms in the order of ϕ^2 may be enough for representation of these functions.

Then we have

$$\frac{\cosh\left[\phi\sqrt{m^2-1}\left(\frac{2}{\pi}\psi-1\right)\right]}{\cosh\left(\phi\sqrt{m^2-1}\right)} \simeq 1+\frac{b^2}{2a^2}(m^2-1)\psi(\psi-\pi)$$

$$\frac{\sinh\left[\phi\sqrt{m^2-1}\left(\frac{2}{\pi}\psi-1\right)\right]}{\cosh\left(\phi\sqrt{m^2-1}\right)} \simeq \frac{b}{a}\sqrt{m^2-1}\left(\psi-\frac{\pi}{2}\right). \quad (19)$$

Equation (19) is valid for $\phi\sqrt{m^2-1} \ll 1$. Then we take summations in (14) up to $m = m'$ after substitution of (19), where the value of m' is chosen so that the following expansions may hold:

$$\frac{4}{\pi}\sum_{m=0,2,\cdots}^{m'}\frac{\epsilon_m \cos m\theta}{(m^2-1)^2} \simeq \sin\theta - \left(\theta-\frac{\pi}{2}\right)\cos\theta$$

$$\frac{2}{\pi}\sum_{m=0,2,\cdots}^{m'}\frac{\epsilon_m \cos m\theta}{m^2-1} \simeq -\sin\theta$$

$$\frac{8}{\pi}\sum_{m=2,4,\cdots}^{m'}\frac{m \sin m\theta}{(m^2-1)^2} \simeq -\left(\theta-\frac{\pi}{2}\right)\sin\theta$$

$$\frac{4}{\pi}\sum_{m=2,4,\cdots}^{m'}\frac{m \sin m\theta}{m^2-1} \simeq \cos\theta. \tag{20}$$

Thus under these approximations, we obtain the following simplified expressions corresponding to (14):

$$\Phi_\xi^{(1)} \simeq -j\beta_g\left[Y^2-\left(\frac{b}{2}\right)^2\right]\sin\theta$$

$$\Phi_\eta^{(1)} \simeq -j\beta_g XY\sin\theta$$

$$\Phi_\zeta^{(1)} \simeq \frac{\pi}{a}\left[Y^2-\left(\frac{b}{2}\right)^2\right]\cos\theta. \tag{21}$$

If a twisted rectangular waveguide of a very flat cross section ($a \gg b$) is of interest, (21) gives a good approximation for the dominant-mode fields in such a structure.

Therefore, we consider a limiting case where the wide-wall dimension a tends to infinity. Then, from (11) and (13), we obtain

$$\beta_g = k, \qquad \theta = \frac{\pi}{2}. \tag{22}$$

Substitution of (22) into (21) gives

$$\Phi_\xi^{(1)} = jk\left[\left(\frac{b}{2}\right)^2 - Y^2\right]$$

$$\Phi_\eta^{(1)} = -jkXY$$

$$\Phi_\zeta^{(1)} = 0. \tag{23}$$

Introducing (23) into (10) and (15), we finally obtain the analytic expressions for the dominant-mode fields in the twisted strip line as follows:

$$E_\xi = j\alpha k\left[\left(\frac{b}{2}\right)^2 - Y^2\right]\exp\left[-jk(Z+\alpha XY)\right]$$

$$E_\eta = \exp\left[-jk(Z+\alpha XY)\right]$$

$$E_\zeta = 0$$

$$H_\xi = -\frac{1}{\zeta_0}E_\eta$$

$$H_\eta = \frac{1}{\zeta_0}E_\xi$$

$$H_\zeta = 0 \tag{24}$$

where the terms in the order of α^2 are neglected and

$$\zeta_0 = \sqrt{\frac{\mu}{\epsilon}}. \tag{25}$$

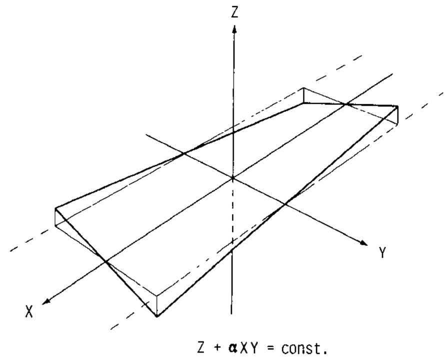

Fig. 5. A propeller-like equiphase surface for the hybrid-mode in a twisted strip line.

In these expressions, the electric and the magnetic fields are represented in terms of the covariant components, respectively.

Since

$$\boldsymbol{a}_\xi \cdot \boldsymbol{a}_\eta = -\alpha^2 XY \tag{26}$$

it is understood that E_ξ and E_η (hence, H_ξ and H_η) are perpendicular to each other in the sense of the first-order approximation.

From (24), the equation for equiphase surfaces of the waves propagating in a positive Z-direction is given by

$$Z + \alpha XY = \text{constant}. \tag{27}$$

One example of a propeller-like equiphase surface is illustratively shown in Fig. 5.

VI. Conclusions

An exact formulation of the boundary conditions has been given for the perfectly conducting surfaces in a helicoidal shape. By using these rigorous boundary conditions, the first-order perturbational solutions have been obtained for the dominant hybrid-mode fields.

The electric and the magnetic-field patterns have been studied by means of modal analysis. It is confirmed that the simultaneous rotation of the field pattern along the twisted surface is predominantly seen for the fundamental TE_{10}-modal component.

Special attention has been given to the backward wave property of the cross-polarized TE_{01} component. It is shown that the modal power for a TE_{01} component vanishes at a certain frequency within the available range. This means that satisfactory matching between the twisted and the straight waveguides would be expected in the neighborhood of this frequency.

The fields in a twisted strip line have also been discussed as the limiting case of the rectangular waveguide. One can easily grasp the physical aspect of the hybrid-mode field by the aid of the propeller-like equiphase surface presented in this paper.

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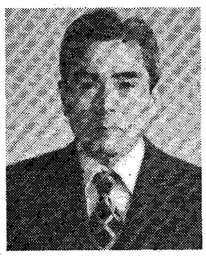

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2–20-GHz GaAs Traveling-Wave Amplifier

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Abstract—**Single-stage and two-stage GaAs traveling-wave amplifiers operating with flat gain responses in the 2–20-GHz frequency range are described. The circuits are realized in monolithic form on a 0.1-mm GaAs substrate with 50-Ω input and output lines. Complete gate and drain dc bias circuitry is included on the chip. By cascading these amplifier chips, a 30-dB gain in the 2–20-GHz range is demonstrated, with 9 ± 1-dB noise figure.**

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I. Introduction

In distributed or traveling-wave amplifiers, the input and output capacitances of electron tubes or transistors are combined with inductors to form two lumped-element artificial transmission lines. These lines are coupled by the transconductance of the active devices [1]–[4].

In this work, we describe one-stage and two-stage traveling-wave amplifiers which operate in the 2–20-GHz frequency range [5]. The amplifiers are truly distributed; the gate and drain lines are two microstrip transmission lines loaded periodically by GaAs FET cells. The basic

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